Calculus 4 (Applications in Economics and Management)

Mock Quiz 2021/06/10 5:30 $\sim$ 6:15

1. Consider the following optimization problem and answer the given questions. Maximize $f(x, y) = x + y$ subject to $x^2 + 2y^2 \leq 6$, $x \geq 0$, $y \geq 0$.

Sol.)

(a) Lagrangian function:

$$\begin{aligned} L(x, y, \lambda_1, \lambda_2, \lambda_3) &= (x + y) - \lambda_1(x^2 + 2y^2 - 6) - \lambda_2(-x) - \lambda_3(-y) \\ &= x + y - \lambda_1(x^2 + 2y^2 - 6) + \lambda_2 x + \lambda_3 y \end{aligned}$$

Kuhn-Tucker Lagrangian function:

$$\tilde{L}(x, y, \lambda) = (x + y) - \lambda(x^2 + 2y^2 - 6)$$

(b) Let $g_1(x, y) = x^2 + 2y^2$, $g_2(x, y) = -x$, $g_3(x, y) = -y$. Then we consider
$J(x, y) = \begin{pmatrix} \nabla g_1 \\ \nabla g_2 \\ \nabla g_3 \end{pmatrix} = \begin{pmatrix} 2x & 4y \\ -1 & 0 \\ 0 & -1 \end{pmatrix}$

If $(x, y) = (0, 0)$, then we consider $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ which is full rank.

On the other hand, if x and y are not all zero, then we are considering $\begin{pmatrix} 2x & 4y \\ -1 & 0 \end{pmatrix}$ when $y \neq 0$, or $\begin{pmatrix} 2x & 4y \\ 0 & -1 \end{pmatrix}$ when $x \neq 0$, which are both full rank.

So there are no points that would fail the NDCQ condition.

(c) The first order conditions:

$$\begin{cases} \frac{\partial L}{\partial x} = 1 - \lambda_1(2x) + \lambda_2 = 0 \\ \frac{\partial L}{\partial x} = 1 - \lambda_1(4y) + \lambda_3 = 0 \\ \lambda_1(x^2 + 2y^2 - 6) = 0 \\ \lambda_2 x = 0 \\ \lambda_3 y = 0 \\ \lambda_1 \geq 0, \quad \lambda_2 \geq 0, \quad \lambda_3 \geq 0 \\ x^2 + 2y^2 \leq 6, \quad x \geq 0, \quad y \geq 0. \end{cases} .$$

(d) The Kuhn-Tucker first order conditions:

$$\begin{cases} \frac{\partial \tilde{L}}{\partial x} = 1 - \lambda(2x) \leq 0 \\ \frac{\partial \tilde{L}}{\partial y} = 1 - \lambda(4y) \leq 0 \\ \frac{\partial \tilde{L}}{\partial \lambda} = (x^2 + 2y^2 - 6) \geq 0 \\ x(1 - \lambda(2x)) = 0 \\ y(1 - \lambda(4y)) = 0 \\ \lambda(x^2 + 2y^2 - 6) = 0 \end{cases}.$$

(e) Plug $(x, y) = (2, 1)$ into (c):

Since $x \neq 0 \Rightarrow \lambda_2 = 0$, $y \neq 0 \Rightarrow \lambda_3 = 0$.

Then use first equation, we know $\lambda_1 = \frac{1}{4}$

(f) Since only the constraint $x^2 + 2y^2 \leq 6$ is binding at $(2, 1)$, the Hessian matrix is

$$H = \begin{pmatrix} 0 & 2x & 4y \\ 2x & -2\lambda_1 & 0 \\ 4y & 0 & -4\lambda_1 \end{pmatrix}, \text{ then } H(2, 1, \tfrac{1}{4}, 0, 0) = \begin{pmatrix} 0 & 4 & 4 \\ 4 & -\frac{1}{2} & 0 \\ 4 & 0 & -1 \end{pmatrix}.$$

We need to check the last $2 - 1$ leading principal minors because we now have 2 variables and 1 constraint, which is $\det \begin{pmatrix} 0 & 4 & 4 \\ 4 & -\frac{1}{2} & 0 \\ 4 & 0 & -1 \end{pmatrix} = 24 > 0$, thus $(x, y) = (2, 1)$ is a local max.

(g) Let $\begin{cases} f(x, y) = x + y \\ g_1(x, y) = x^2 + 2y^2 \leq 6 = a_1 \\ g_2(x, y) = -x \leq 0 = a_2 \\ g_3(x, y) = -y \leq 0 = a_3 \end{cases}$

If we change the constraint such that $a_1 = 5.9$, $a_2 = 0.2$, $a_3 = 0.3$, then according to

$$\lambda_j(a_1, a_2, a_3) = \frac{\partial}{\partial a_j} f(x(a_1, a_2, a_3), y(a_1, a_2, a_3))$$

We will have

$$\begin{aligned} & f(x(5.9, 0.2, 0.3), y(5.9, 0.2, 0.3)) \\ & \quad \approx f(x(6, 0, 0), y(6, 0, 0)) + \lambda_1(5.9 - 6) + \lambda_2(0.2 - 0) + \lambda_3(0.3 - 0) \\ & \quad = f(2, 1) + \frac{1}{4}(-0.1) + 0 + 0 \\ & \quad = 3 - 0.025 = 2.975 \end{aligned}$$

(h) The original Lagrangian:

$$L(x, y, \lambda_1, \lambda_2, \lambda_3) = x + y - \lambda_1(x^2 + 2y^2 - 6) + \lambda_2 x + \lambda_3 y$$

Now let:

$$L'(x, y, \lambda_1, \lambda_2, \lambda_3; a_1, a_2, a_3, a_4, a_5) = a_1 x + a_2 y - \lambda_1(a_3 x^2 + a_4 y^2 - a_5) + \lambda_2 x + \lambda_3 y$$

Calculate

$$\begin{cases} \frac{\partial L'}{\partial a_1} = x \\ \frac{\partial L'}{\partial a_2} = y \\ \frac{\partial L'}{\partial a_3} = -\lambda_1 x^2 \\ \frac{\partial L'}{\partial a_4} = -\lambda_1 y^2 \\ \frac{\partial L'}{\partial a_5} = \lambda_1 \end{cases}$$

By the Envelope Theorem,

The approximation of the maximum is

$$f(x,y) + \frac{\partial L'}{\partial a_1}(0.9-1) + \frac{\partial L'}{\partial a_2}(1.1-1) + \frac{\partial L'}{\partial a_3}(1.1-1) + \frac{\partial L'}{\partial a_4}(2.1-2) + \frac{\partial L'}{\partial a_5}(5.9-6)$$

$$= 3 + 2(-0.1) + 1(0.1) - 1(0.1) - \frac{1}{4}(0.1) + \frac{1}{4}(-0.1)$$

$$= 2.75$$